Universality classes for self-similarity of noiseless multidimensional Burgers turbulence and interface growth

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The present work is devoted to the evolution of random solutions of the unforced Burgers and KPZ equations in d dimensions in the limit of vanishing viscosity. We consider a cellular model and as initial condition assign a value for the velocity potential chosen independently within each cell. We show that the asymptotic behavior of the turbulence at large times is determined by the tail of the initial potential probability distribution function. Three classes of initial distribution leading to self-similar evolution are identified: (a) distributions with a power-law tail, (b) compactly supported potential, (c) stretched exponential tails. In class (c) we find that the mean potential (mean height of the surface) increases logarithmically with time and the "turbulence energy" $E(t) = \langle \mathbf{v}^2(\mathbf{x},t) \rangle = \langle (\boldsymbol{\nabla} \psi(\mathbf{x},t))^2 \rangle$ (mean square gradient of the surface) decays as t^{-1} times a logarithmic correction. In classes (a) and (b) we find that the changes in the mean potential and energy have a power-law time dependence, namely, $E(t) \propto t^{-p}$ where the index p lies in the interval $2 > p > (2 - d)/2$. In class (c) the roughness of the surface, measured by its mean-square gradient, may either decrease or increase with time. We discuss also the influence of finite viscosity and long range correlation on the late stage evolution of the Burgers turbulence.

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I. INTRODUCTION

The multidimensional Burgers equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v}) = \nu \Delta \mathbf{v}, \qquad (1)$$

is a generalization of the well-known Burgers equation

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \nu \frac{\partial^2 v}{\partial x^2}. \qquad (2)$$

The nonlinear diffusion equation (2), which was originally introduced by J. Burgers in 1939 as a model of hydrodynamical turbulence [1,2], was later shown to describe a variety of nonlinear wave phenomena arising in the theory of wave propagation, acoustics, plasma physics (see, e.g., Ref. [3–5]).

The Burgers equation (2) describes two principal effects inherent in any turbulence [6]: the nonlinear redistribution of energy over the spectrum and the action of viscosity in small-scale regions. Although external forces are not present in Eq. (2), the one-dimensional Burgers equation does describe the decay of turbulence, i.e., the nonlinear transformation of the random initial field $v_0(x)$. The Burgers equation shares a number of properties with the Navier-Stokes equation, namely, the same type of nonlinearity, of invariance groups, of the energy-dissipation relation, of the existence of a multidimensional version, etc. However, Burgers equation is integrable and therefore is not sensitively dependent on initial conditions. The differences between the Burgers and

Navier-Stokes equations are as interesting as the similarities [7] and this is also true for the multidimensional Burgers equation (1).

The three-dimensional form of Eq. (1) has been used to model the formation of the large scale structure of the Universe when pressure is negligible, that is, during the nonlinear stage of the gravitational instability arising from random initial perturbations [8–11]. Other problems leading to the multidimensional Burgers equation, or variants of it, include surface growth under sputter deposition and flame front motion [12]. In such instances, the potential ψ corresponds to the shape of the front's surface, and the equation for the velocity potential ψ is identical to the KPZ (Kardar, Parisi, Zhang) equation [12–14]. For the deposition problem $\mathbf{v} = -\boldsymbol{\nabla} \psi$ is the gradient of the surface. The roughness of the surface, measured by its mean-square gradient $E(t) = \langle (\boldsymbol{\nabla} \psi(\mathbf{x},t))^2 \rangle = \langle \mathbf{v}^2(\mathbf{x},t) \rangle$, may either decrease or increase with time. Nevertheless we will use the expression "turbulence energy" for this value of $E(t)$. With external random forces the Burgers and KPZ equations describe phenomena such as turbulence without pressure, disordered systems, directed polymers, etc. [12,15–18].

Here we will consider the evolution of the velocity field $\mathbf{v}(\mathbf{r},t)$ and potential $\psi(\mathbf{r},t)$ as given by the noiseless multidimensional Burgers and KPZ equations. In this case the evolution of the fields is fully determined by the statistical properties of the initial field $\mathbf{v}_0(\mathbf{r})$, which is assumed to be potential $\mathbf{v}_0(\mathbf{r}) = -\boldsymbol{\nabla} \psi_0(\mathbf{r})$. Our main attention will be given to the case of vanishing viscosity ($\nu \to 0$), when the dissipative effects are important only in the vicinity of shocks.

It is known that the asymptotic behavior ($t \to \infty$) of Burgers turbulence strongly depends on the behavior of the structure function of the initial potential $d_\psi(\rho) = \langle [\psi_0(\mathbf{r} + \rho) - \psi(\mathbf{r})]^2 \rangle$ at large distances [2,5]. If the structure func-

tion $d_\psi(\rho)$ increases as a power law in space then the initial potential field is Brownian, or fractional Brownian motion, and some scaling may be used [2,5,11,19]. In this case the turbulence is self-similar. The evolution of the external scale $L(t)$ of the self-similar solution in time is determined by the spatial behavior of $d_\psi(\rho)$ at large distances. Very recently a complete solution of the one-dimensional Burgers equation, with initial Gaussian white noise distributed data in the inviscid limit, has been obtained [20].

If the structure function of the potential is bounded at $\rho \to \infty$, then scaling arguments no longer work. Kida [21] has proposed using, for the one-dimensional Burgers turbulence, a discrete cellular model of the initial conditions with independent distribution of potential in different cells. He has shown that the energy decays as t^{-1} with a logarithmic correction when the probability distribution of the potential in each cell has a stretched exponential tail $\sim \exp(-H^\beta)$. Later it was shown [22–24] that for an initial continuous Gaussian field all of the statistical characteristics of one-dimensional Burgers turbulence become self-similar and the energy decays as $t^{-1}/(\ln t)^{1/2}$. Several models of the evolution of Burgers turbulence with an initial perturbation of non-Gaussian type have also been proposed (Ref. [14], and references therein, [25–28,30]). It has been shown that the law of energy decay strongly depends on the statistical properties of the initial field with homogeneous potential.

It is known that in the limit of vanishing viscosity the solution of Burgers' equation is reduced to searching for the absolute maximum of some function of the initial potential [31]. Consequently, the statistical properties of Burgers' turbulence are determined by the behavior of the probability distribution of the initial potential. One of the important results of the classical theory of extrema is that there are only three universal classes of ''extreme value distributions'' of the sequence of independent and identically distributed random variables [32]. This result was used in Ref. [33] for dealing with the problem of the equilibrium of low-temperature physics of disordered systems and partly for the one-dimensional decaying Burgers turbulence. One of our main tasks is to show that, in the discrete cellular model of the initial condition for the d-dimensional Burgers turbulence, there are also three classes of universal self-similar evolution of the velocity and potential fields.

The paper is organized as follows. In Sec. II we formulate our problem and list some elementary results about the Burgers equation. We also consider the evolution of potential and velocity fields for a simple kind of perturbation in individual cells at the initial stage. In Sec. III we derive the general expression for the energy and probability distribution functions of the potential and vector velocity fields. In Sec. IV we show that three types of initial condition lead to asymptotic self-similar behavior of Burgers' turbulence. Sec. V presents concluding remarks. We also discuss the influence of finite viscosity and long range correlation on the late stage evolution of Burgers' turbulence.

II. DESCRIPTION OF THE BASIC MODEL

A. Basic equations and local self-similarity

We will discuss the initial-value problem for the unforced Burgers equation (1), and consider only the potential solution of this equation, namely,

$$\mathbf{v}(\mathbf{x},t) = -\nabla \psi(\mathbf{x},t). \tag{3}$$

The velocity potential $\psi(\mathbf{x},t)$ satisfies the following nonlinear equation:

$$\partial_t \psi = \frac{1}{2}(\nabla \psi)^2 + \nu \Delta \psi. \tag{4}$$

This equation is the same as the KPZ equation [12–14], which is usually written in terms of the variable $h = \lambda^{-1}\psi$. The parameter λ has the dimensions of length divided by time and is the local velocity of the surface growth. Henceforth h has the dimensionality of length and is the measure of shape of the surface. Using the Hopf-Cole transformation $\psi = \ln U$ [31,34], one can reduce Eq. (4) to a linear diffusion equation. We are mainly interested here by solutions in the limit $\nu \to 0$. Use of Laplace's method then leads to the following ''maximum representation'' for the potential and velocity fields in the limit of vanishing viscosity [31,5,11]:

$$\psi(\mathbf{x},t) = \max_{\mathbf{y}} \Phi(\mathbf{x},\mathbf{y},t), \tag{5}$$

$$\Phi(\mathbf{x},\mathbf{y},t) = \psi_0(\mathbf{y}) - \frac{(\mathbf{x}-\mathbf{y})^2}{2t}, \tag{6}$$

$$\mathbf{v}(\mathbf{x},t) = \frac{\mathbf{x}-\mathbf{y}(\mathbf{x},t)}{t}. \tag{7}$$

Here $\psi_0(\mathbf{y})$ is the initial potential and $\mathbf{v}_0(\mathbf{x}) = -\nabla \psi_0(\mathbf{x})$. In Eq. (7) $\mathbf{y}(\mathbf{x},t)$ is the Lagrangian coordinate where the function $\Phi(\mathbf{x},\mathbf{y},t)$ achieves its global or absolute maximum for a given coordinate $\mathbf{x}$ and time t.

At large times the paraboloid peak in Eq. (6) defines a much smoother function than the initial potential $\psi_0(\mathbf{y})$. Consequently, the absolute maximum of $\Phi(\mathbf{x},\mathbf{y},t)$ coincides with one of the local maxima of $\psi_0(\mathbf{y})$. The Lagrangian coordinate $\mathbf{y}(\mathbf{x},t)$ then becomes a discontinuous function of $\mathbf{x}$, constant within a cell, but jumping at the boundaries [5,11]. In each cell fluid particles move away from the small region near the cell center $\mathbf{y}_k$. The velocity field $\mathbf{v}(\mathbf{x},t)$ has discontinuities (shocks) and the potential field $\psi(\mathbf{x},t)$ has gradient discontinuities (cusps) at the cell boundaries; these shock surfaces or walls form a connected structure. Inside the cells the velocity and potential fields (surface shape) have a universal self-similar structure.

$$\mathbf{v}(\mathbf{x},t) = \frac{\mathbf{x}-\mathbf{y}_k}{t}, \tag{8}$$

$$\psi(\mathbf{x},t) = \psi_0(\mathbf{y}_k) - \frac{(\mathbf{x}-\mathbf{y}_k)^2}{2t}. \tag{9}$$

The longitudinal component of the velocity vector $\mathbf{v}(\mathbf{x},t)$ consists of a sequence of sawtooth pulses with random positions of the shocks and ''zeros,'' just as in one dimension. The transverse components consist of sequences of rectangular pulses with random amplitudes and random positions of the shocks. Wall motion results in continuous change of cell shape with cells swallowing their neighbors and thereby inducing growth of the external scale $L(t)$ of the Burgers tur-

bulence. At large times the behavior of the turbulence will be determined by the statistical properties of the initial potential and, moreover, by the statistical properties of local maxima $\psi_0(\mathbf{y}_k)$.

B. The cellular model and initial stage of evolution

We consider the cellular model of the initial conditions, in which we assume that space is divided into identical cells, each having a volume $dV = L_0^d$, where L_0 is the length of the cell and d is the spatial dimension. We assign an initial value for the potential which is chosen independently within each cell. The same approach was used in Ref. [21] in order to find the probability distributions of the amplitudes and velocities of shocks for the one-dimensional Burgers turbulence and in Refs. [5,8] for multidimensional Burgers turbulence in the case where the initial potential had a a stretched exponential tail. In the ''shot-noise model'' [25] it is assumed that the initial potential is a sum of potentials of ''unit nonhomogeneities'' with random amplitudes and scales and with a Poisson ensemble for the position.

First we consider the evolution of the velocity and potential fields of the ''unit nonhomogeneities'' inside the cells assuming that there is no interaction between cells. We assume for simplicity that inside each cell the initial potential and velocity are isotropic and

$$\psi_0(\mathbf{x}) = \psi_0(1 - \mathbf{x}^2/l_0^2), \quad |\mathbf{x}| < l_0, \tag{10}$$

$$\mathbf{v}_0(\mathbf{x}) = V_0\mathbf{x}/l_0, \quad |\mathbf{x}| < l_0, \ V_0 = 2\psi_0/l_0, \tag{11}$$

where $\psi_0(\mathbf{x}) = 0, \mathbf{v}_0(\mathbf{x}) = 0$ when $|\mathbf{x}| > l_0$ and V_0 is the amplitude of the velocity at the border $|\mathbf{x}| = l_0$. From the solution (6), (7) we have for the velocity field

$$\mathbf{v}(\mathbf{x},t) = V_0 \frac{\mathbf{x}}{l_0(1 + V_0t/l_0)}, \quad |\mathbf{x}| < x_s, \tag{12}$$

where x_s is the position of the shock surface

$$x_s = l_0(1 + V_0t/l_0)^{1/2} = (l_0^2 + 2\psi_0t)^{1/2}. \tag{13}$$

In the one-dimensional case this solution is a well known N wave [3].

The energy of the velocity field is given by

$$E(t) = \int v^2(\mathbf{x},t)d^d\mathbf{x} = \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^{x_s} v^2 r^{d-1}dr, \tag{14}$$

and from Eqs. (12),(13) it follows that

$$E(t) = E_0(1 + V_0t/l_0)^{(d-2)/2}, \quad E_0 = \frac{2\pi^{d/2}V_0^2l_0^d}{\Gamma(d/2)(d+2)}, \tag{15}$$

where $\Gamma(z)$ is a gamma function. From Eq. (15) we see that for $d=1$ the energy $E(t)$ decreases with time, for $d=2$ $E(t)$ is constant, and for $d \geqslant 3$ $E(t)$ increases with time. It is possible to show that $E(t)$ may increase with time when $d \geqslant 3$ even if the viscosity coefficient ν is finite. The increase of the energy in the multidimensional Burgers equation (in contrast with $d=1$) is the result of this equation not having a conservation form.

Moreover it easy to show that in the multidimensional Burgers equation the energy does not conserve in the limit of vanishing viscosity even at the initial stage of evolution, when the velocity field does not have shocks. Let us consider the evolution of the continuos velocity field having only a radial component $v_r(\mathbf{x},t) = v(r,t)$. In the limit of vanishing viscosity we have the implicit solution of Burgers equation $v = v_0(r - vt)$. Substitute this solution into Eq. (14) and replace the Eulerian coordinate r by Lagrangian coordinate $y: r = y + tv_0(y)$, we have the following expression for the energy:

$$E(t) = \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty v_0^2(y)[y + tv_0(y)]^{d-1}\left(1 + t\frac{\partial v_0(y)}{\partial y}\right)dy. \tag{16}$$

From Eq. (16) we see that the energy $E(t)$ of continuous field conserves only in the one-dimensional case ($d=1$). In particular for $d=2$ we have from Eq. (16)

$$E(t) = 2\pi\int_0^\infty v_0^2(y)ydy + t\frac{4\pi}{3}\int_0^\infty v_0^3(y)dy. \tag{17}$$

Thus for the positive perturbation of the initial velocity the energy (mean-square gradient of the surface) increases with time, and for the negative perturbation of the velocity the energy decreases with time. This effect takes place due to the relatively more significant contribution into mean-square gradient of the surface the regions with large distances r (16).

The asymptotic behavior of the potential and velocity fields is determined only by the positions and values of the maximum of the initial potential in the cells (8), (9). For relatively large time ($V_0t \gg l_0$) the velocity inside the cells has the universal behavior

$$\mathbf{v}(\mathbf{x},t) = \mathbf{x}/t, \quad |\mathbf{x}| \leqslant x_s = (2\psi_0t)^{1/2}. \tag{18}$$

The position of the shock surface x_s and the energy of the field

$$E(t) \sim \psi_0^{(d+2)/2}t^{(d-2)/2} \tag{19}$$

are determined only by the value of the initial potential maximum ψ_0. It is easy to show that this asymptotic behavior takes place for the arbitrary initial localized perturbations with the potential maximum ψ_0.

Let us assume ($L_0 \gg l_0$) and that the maximum value in the ''ith'' cell is $\psi_{0,i}$. The interaction between cells begins when the shock surfaces of neighboring cells come in contact, i.e., when $|x_{s,i}| + |x_{s,j}| = L_0$. At rather large time the border between the cells becomes a hyperplane, moving with constant velocity $|\psi_{0,i} - \psi_{0,j}|/|\mathbf{y}_i - \mathbf{y}_j|$ away from the cell with a larger value of the unital potential [5]. When the potential maximum in all of the cells is the same, i.e., when $\psi_{0,i} = \psi_0$, then at large times the hypersurfaces between neighboring cells are immobile and the velocity $\mathbf{v}(\mathbf{x},t)$ and potential $\psi(\mathbf{x},t)$ fields have stationary periodic structure. The gradient of the velocity field decreases as t^{-1}, which means that the energy density $E(t) = \langle\mathbf{v}^2\rangle$ decreases according to

$$E(t) \sim L_0^2/t^2. \tag{20}$$

For a random distribution of the potential maximum $\psi_{0,i}$ we have the permanent absorption of cells with low values of the potential so that the external scale of the turbulence $L(t)$ increases with time. The mean energy evolution strongly depends on the statistical properties of the maximum potential distribution. The energy decay is slower than for the periodic structure (20). The interaction between cells changes also the law (19). Moreover, for $d \geqslant 3$, due to the interaction between cells the increase of energy $E(t)$ Eq. (19) may change to a to a decrease. It will be shown later that asymptotically $E(t) \propto t^{-p}$ with p lyings in the interval $2 > p > (2-d)/2$.

III. ENERGY EVOLUTION AND PROBABILITY DISTRIBUTIONS OF THE TURBULENCE AND THE INTERFACE

We assume that the initial random potential $\psi_0(\mathbf{y})$ is statistically homogeneous. Then, from Eq. (4) it follows that the turbulence energy is determined by the time derivative of the mean potential

$$E(t) = \langle \mathbf{v}^2(\mathbf{x},t) \rangle = \langle [\nabla \psi(\mathbf{x},t)]^2 \rangle = 2 \frac{\partial \langle \psi(\mathbf{x},t) \rangle}{\partial t}. \quad (21)$$

Let $P_\psi(H,t)$ denote the probability density of ψ at time t and let $Q_\psi(H,t)$ denote the cumulative probability to have $\psi < H$, given by

$$Q_\psi(H,t) = \int_{-\infty}^H P_\psi(A,t) dA. \quad (22)$$

The mean value of the potential $\langle \psi \rangle$ (i.e., mean value of the height of the surface) at time t is then expressible as

$$\langle \psi \rangle = \int_{-\infty}^{+\infty} H P_\psi(H,t) dH. \quad (23)$$

Thus we can find the mean energy of the turbulence if we know the probability distribution function $P_\psi(H,t)$, where $P_\psi(H,t)dH$ is the probability that the absolute maximum of $\Phi(\mathbf{x},\mathbf{y},t)$ (6) lies in the interval $H, H+dH$ for all $\mathbf{y}$.

The function $P_{\psi,\mathbf{y}}(H,\mathbf{y};\mathbf{x},t)$ is the joint probability distribution function (PDF) of ψ and $\mathbf{y}$ and $P_\psi dH d^d \mathbf{y}$ is the the probability that the absolute maximum of $\Phi(\mathbf{x},\mathbf{y},t)$ lies in the interval $H, H+dH$ in the volume element $d^d \mathbf{y}$. By integrating the $P_{\psi,\mathbf{y}}(H,\mathbf{y};\mathbf{x},t)$ over H we obtain the PDF of the Lagrangian coordinate $\mathbf{y}$:

$$P_\mathbf{y}(\mathbf{y};\mathbf{x},t) = \int_{-\infty}^{+\infty} P_{\psi,\mathbf{y}}(H,\mathbf{y};\mathbf{x},t) dH, \quad (24)$$

which permits one to find the probability distribution function of the velocity field (7).

Since the initial potential is homogeneous we will consider the point $\mathbf{x}=0$ and no longer include $\mathbf{x}$ in the parameter of the probability distribution function. In the ''cellular'' model we assume that the space is divided up into elementary cells of volume $dV = L_0^d$. We consider the late stage of evolution when the external scale $L(t)$ is much greater than

L_0. On this scale the initial potential field is homogeneous. (Formally we can introduce a random uniform distribution of cell positions.)

The asymptotic behavior of the probability distribution function of absolute maximum of a large number of random quantities is determined by the asymptotic properties of their cumulative distribution function at large values. We assume that in each cell the cumulative probability function of the initial potential $\psi_0(\mathbf{y})$ is the same and can be represented in the following form:

$$\mathrm{Prob}(\psi_0 < H) = F(H) = 1 - f(H). \quad (25)$$

The cumulative probability distribution function for Φ [Eq. (6)] in the ith cell is

$$F_i(H) = 1 - f_i(H), \quad f_i(H) = f\left(H + \frac{\mathbf{y}_i^2}{2t}\right). \quad (26)$$

Here $\mathbf{y}_i$ is the coordinate of the ith cell. Due to the independence of the initial potential in different cells we can determine the cumulative distribution function ψ as

$$\mathrm{Prob}(\psi < H) = Q(H,t) = \prod_i F_i(H)$$

$$\equiv \prod_i [1 - f_i(H)]. \quad (27)$$

From Eq. (26) we see that $f_i(H)$ decreases with the distance $|\mathbf{y}|$.

We will consider the limit when t becomes large, so that the number of factors in Eq. (27) becomes significant. This means that the absolute maximum of Φ is reached for rather large H, so in each cell we have $f_i(H) \ll 1$, thus $1 - f_i(H) \simeq \exp[-f_i(H)]$, and we can rewrite Eq. (27) in the form

$$Q(H,t) = e^{-N(H,t)}, \quad (28)$$

where

$$N(H,t) = \sum_i f_i(H) = \sum_i f\left(H + \frac{\mathbf{y}_i^2}{2t}\right). \quad (29)$$

At large t when the difference between f_i in neighboring cells is small, we can replace the summation in Eq. (29) by an a volume integral:

$$N(H,t) = \frac{1}{L_0^d} \int f\left(H + \frac{\mathbf{y}^2}{2t}\right) d^d \mathbf{y}. \quad (30)$$

For large H the events for which ψ_i is greater than H are rare and they have a Poisson distribution. Thus N in Eq. (28) is a mean number of events when $\psi > H$.

From (28,30) we have two equivalent expressions for the mean potential:

$$\langle \psi \rangle = \int_{-\infty}^{+\infty} H \frac{\partial}{\partial H} e^{-N(H,t)} dH = \int_0^{+\infty} H(\Theta,t) e^{-\Theta} d\Theta, \quad (31)$$

where $H(\Theta,t)$ is a solution of the equation

$$N(H,t)=\Theta. \tag{32}$$

Formally we can integrate in Eq. (31) over the infinite interval of H since the probability of H being negative is negligibly small when t is large.

In order to find the probability distribution function of the velocity we first of all need to find the joint probability distribution of value and the coordinate of the absolute maximum of Φ. The probability that the absolute maximum is in the ith cell, with a value in the interval $H, H+dH$, is

$$\text{Prob}(\mathbf{y}\in\Delta_j, H\in[H,H+dH])$$

$$= -\frac{\partial}{\partial H}f_j(H)\prod_{i\neq j}[1-f_i(H)]. \tag{33}$$

Here the product over all cells with $i\neq j$ is the probability that outside the jth cell all local maxima are less than H.

With the same assumption as before we can find the joint probability distribution coordinate and value of the absolute maximum. At large t we have

$$P_y(\mathbf{y},t)=P(\mathbf{y}\in\Delta_j)/L_0^d \tag{34}$$

and after integration over H we have the following expression for the probability distribution function of the absolute maximum coordinate

$$P_y(\mathbf{y},t)=-\frac{1}{L_0^d}\int_{-\infty}^{+\infty}\frac{\partial}{\partial H}f\left(H+\frac{\mathbf{y}^2}{2t}\right)e^{-N(H,t)}dH. \tag{35}$$

Integration by parts leads to

$$P_y(\mathbf{y},t)=\frac{1}{L_0^d}\int_{-\infty}^{+\infty}f\left(H+\frac{\mathbf{y}^2}{2t}\right)\frac{\partial}{\partial H}e^{-N(H,t)}dH$$

$$=\frac{1}{L_0^d}\int_0^{+\infty}f\left(H(\Theta)+\frac{\mathbf{y}^2}{2t}\right)e^{-\Theta}d\Theta. \tag{36}$$

Here $H(\Theta)$ is a solution of Eq. (32). Using Eq. (30) it is easy to see that the norm of $P_y(\mathbf{y},t)$ is unity.

In the theory of the absolute maximum of random sequences [32] the large parameter of the theory is M — the number of ''points'' in a sequence. In our case we consider formally an infinite number of points, but owing to the parabolic form used in Eq. (29) the effective number of cells is finite and increases with time. We stress again that expressions (30), (36) are valid only when $L(t)\gg L_0$.

IV. THREE TYPES OF SELF-SIMILARITY FOR BURGERS TURBULENCE

In a previous section we derived expressions for the probability distribution function of the potential $\psi(\mathbf{x},t)$ (28), (30) and for the mean potential (31) which determines the energy evolution of the turbulence (21). We also derived the form of the probability distribution function for the absolute maximum coordinate (35) which coincides with the form of the velocity probability distribution function (7). It is known that there are three types, known as the Frechet, Weibull, and Cumbel classes, of asymptotic behavior for the probability distribution function of the large sequence of random quantities [32,33]. In this section we will show that these three types of initial condition lead to three types of asymptotic self-similar behavior of the Burgers turbulence.

A. Distributions with a power-law tail: Frechet class

Let us now assume that the cumulative probability distribution function of the initial potential (25) has a power law tail

$$f(H)=\left(\frac{H_*}{H}\right)^{\gamma}. \tag{37}$$

In the integral (30) for $N(H,t)$ we can integrate over the radius $r=|\mathbf{y}|$ and obtain

$$N(H,t)=\frac{2\pi^{d/2}}{\Gamma(d/2)}\left(\frac{H_*}{H}\right)^{\gamma}\int_0^{\infty}\frac{r^{d-1}dr}{\left(1+\dfrac{r^2}{2Ht}\right)^{\gamma}}. \tag{38}$$

This integral converges when $\gamma>d/2$, and gives the asymptotic distribution of the potential ψ when the probability distribution function of the initial potential decays sufficiently rapidly. Replacing the variable r by $r=(2tH)^{1/2}x$ allows us to write $N(H,t)$ as a power-law function of H

$$N(H,t)=\frac{H_*^{\gamma}}{L_0^d}\frac{t^{d/2}}{H^{(2\gamma-d)/2}}C_d,$$

$$C_d=\frac{2^{(d+1)/2}\pi^{d/2}}{\Gamma(d/2)}\int_0^{\infty}\frac{x^{d-1}}{(1+x^2)^{\gamma}}dx, \tag{39}$$

where C_d depends on d as well as γ. Thus we have the self-similar cumulative function (28) of the potential (height of the surface) is given by

$$Q(H)=\exp\left[-\left(\frac{H_*}{H}\right)^{(2\gamma-d)/2}\left(\frac{t}{t_{nl}}\right)^{d/2}\right]. \tag{40}$$

Here we introduce t_{nl} as a nonlinear time of evolution

$$t_{nl}=\frac{L_0^2}{H_*C_d^{2/d}}. \tag{41}$$

It follows from Eq. (40) that the probability distribution function of potential is self-similar with the amplitude scale $H_s(t)$:

$$H_s(t)=H_*\left(\frac{t}{t_{nl}}\right)^{d/(2\gamma-d)}. \tag{42}$$

The time evolution of the velocity variance (21) is determined by the time derivative of the mean potential (31). It is easy to see from Eq. (40) that the expression for the mean potential converges if $\gamma>d/2+1$, and in this case the mean value $\langle\psi\rangle\sim H_s(t)$. For $\gamma<d$ the mean potential $\langle\psi\rangle$ increases faster in time then the local height of the surface which grows proportionally $t\lambda$. The superlinear increase of

mean potential is possible while the KPZ equation (4) describes the evolution of shape $\psi(\mathbf{x},t) \sim h(\mathbf{x},t) = H(\mathbf{x},t) - t\lambda$ of the surface and the normal growing of the surface is excluded a for this equation [12–14].

From Eq. (21) the velocity variance (turbulence energy) is given by

$$E(t) = 2\frac{\partial\langle\psi\rangle}{\partial t} \sim \frac{\partial H_s}{\partial t} \propto t^{-p}; \quad p = \frac{2(\gamma - d)}{2\gamma - d}. \quad (43)$$

We stress here that for this expression to be valid γ must satisfy $\gamma > \gamma_{cr} = d/2 + 1$. From Eq. (43) we see that when γ lies in the interval

$$\frac{d}{2} + 1 < \gamma < d, \quad (44)$$

the velocity variance increases with time. Thus the velocity variance increases only if the spatial dimension d satisfies $d \geq 3$. For $d = 3$ we have the same results as the ''shot-noise model'' of the initial perturbation [25]. On the left hand side of the inequality (44), if $\gamma = d/2 + \varepsilon, (\varepsilon \ll 1)$ we have for the index p in Eq. (43)

$$p_\varepsilon = \frac{2 - d + 2\varepsilon}{2}. \quad (45)$$

This means that (a) when $d = 1$ the energy always decays with $p \simeq 1/2$, (b) when $d = 2$ the variance decays very slowly with $p = \varepsilon$, (c) when $d = 3$ the variance increases with $p \simeq -1/2$, and (d) when $d \gg 1$ then $p \simeq (2 - d)/2$.

This behavior of critical index has a simple explanation if we consider the velocity field evolution in an elementary cell at the stage before the interaction with other cells takes place (Sec. II B). It follows from Eq. (19) that the energy $E(t)$ of an elementary volume varies as $E(t) \sim \psi_0^{(d+2)/2} t^{(d-2)/2}$. When the power index γ of potential probability distribution function is close to the critical value γ_c, then probability distribution function of the initial potential has a relatively slow decay and the general evolution of the turbulence variance is determined by the cell with very high amplitude. It is easy to see that at the critical point the time dependence of the mean variance (45) is the same as the time dependence of the energy of an initial cell (19).

As γ increases the interaction between the elementary cells begins to play a more and more important role. When $\gamma = d$ and $(d > 2)$ the increase of variance $E(t)$ will switch to a decrease with time. When $\gamma \gg d$ the variance will decay as t^{-1} which is close to the law for the stretched exponential type of initial potential (see Sec. IV C).

It is easy to show from Eq. (36) that the probability distribution function of the coordinate of the absolute maximum of the potential is also self-similar, i.e.,

$$W(\mathbf{y}) = \frac{1}{L^d(t)} W_s\left(\frac{\mathbf{y}}{L(t)}\right), \quad (46)$$

where $W_s(z)$ is the dimensionless form of probability distribution function

$$W_s(z) = A\int_0^\infty \frac{e^{-\Theta} d\Theta}{\left[\dfrac{1}{\Theta^{2/(2\gamma - d)}} + z^2\right]^\gamma}, \quad (47)$$

A is normalization coefficient depending on d and γ. In (46) $L(t)$ is the external scale of turbulence and

$$L(t) = \frac{(2C_d)^{1/2}(tH_*)^{\gamma/(2\gamma - d)}}{L_0^{d/(2\gamma - d)}} \sim L_0\left(\frac{t}{t_{nl}}\right)^{\gamma/(2\gamma - d)}, \quad (48)$$

The distribution (47) decays $\sim z^2$ at small distances and by the power law at large distances: $W(z) \simeq A/z^{2\gamma}$. From this we see again that the mean energy is finite when $\gamma > \gamma_{cr} = d/2 + 1$, which is also the condition that the mean energy in separate cells (19) $E(t) \sim \int H^{(d+2)/2} f'(H) dH$ be finite. Note that the mean initial energy (15) $E(0) = \langle E_0\rangle/L_0^d \sim \int H^2 f'(H) dH$ is finite if $\gamma > 2$ for all value of d.

B. Compactly supported potential: Weibull class

Let us now assume that the initial potential is bounded by some value H_m [$\psi_0(y) \leq H_m$ in all cells] and that the cumulative probability distribution function has a power law behavior in the neighborhood of the maximum, i.e.,

$$f(H) = \left(\frac{H_M - H}{H_*}\right)^\alpha, \quad H < H_M \quad (49)$$

and is zero for $H > H_M$. By the same procedure as above (see Sec. IV A) it follows that the cumulative probability distribution function of potential has the form (28) with

$$N(H,t) = \frac{(H_M - H)^{(2\alpha + d)/2}(2t)^{d/2}}{L_0^d H_*^\alpha} C_d,$$

$$C_d = \frac{2^{(d+1)/2}\pi^{d/2}}{\Gamma(d/2)}\int_0^1 r^{d-1}(1 - r^2)^\alpha dr \quad (50)$$

and is self-similar for arbitrary dimension d and power index α. The mean potential tends to the maximum H_M according to the power law

$$\langle\psi\rangle = H_M - \tilde{c}H_*(t_{nl}/t)^{d/(2\alpha + d)}. \quad (51)$$

Here $\tilde{c}$ is a positive numerical coefficient, and t_{nl} is the nonlinear time t_{nl} (41), where the constant C_d is determined by the expression (50).

From Eq. (51) it follows that the energy always decreases according to the power law

$$E(t) = 2\frac{\partial\langle\psi\rangle}{\partial t} \sim t^{-p}, \quad p = \frac{2(\alpha + d)}{2\alpha + d}. \quad (52)$$

It follows from Eqs. (36),(49),(50) that the PDF of the absolute maximum coordinate $\mathbf{y}$ is self-similar [Eq. (46)] with the spatial scale $L(t)$

$$L(t) \sim L_0(t/t_{nl})^{\alpha/(2\alpha + d)}. \quad (53)$$

The general result for the length scale evolution for systems in the Weibull class has actually been noted in Eq. [29] (al-

though not explicitly derived). It is easy to see that the energy (43),(52) and the external scale (48), (53) of the turbulence are described by the same expressions for a distribution having a power-law tail and for a compactly supported distribution if we set $\alpha = -\gamma$ in Eq. (49) ($\gamma < 0$).

The form of the distribution of the absolute maximum coordinate is now defined by the integral

$$W_s(z) = A \int_{\theta_*(z)}^{\infty} (\theta^{2/(2\alpha+d)} - z^2)^{\alpha} e^{-\theta} d\theta,$$

$$\theta_*(z) = z^{2\alpha+d}. \qquad (54)$$

It can be seen that this distribution decays exponentially at large z

$$W_s(z) \sim e^{-z^{2\alpha+d}} z^{\alpha(2-d-2\alpha)}. \qquad (55)$$

It is interesting to compare the evolution of Burgers turbulence for a compactly supported initial potential with the evolution of turbulence for a Gaussian initial potential having scaling properties with dimension h, such that $0 \geqslant h > -1$ [30]. For values of h that satisfy this condition the initial potential correlation function has a singularity at the origin. In Ref. [30] a special class of random solution of the one-dimensional Burgers equation was constructed. For this solution the self-similar behavior is true at all times and not just asymptotically. Such scaling behavior corresponds to the characteristic length $L(t) \sim t^{1/(2-h)}$. Comparing this result with Eq. (53) we see that for $\alpha = -1/h$ and $\alpha > 1$ the evolution of turbulence with a compactly supported initial potential tends asymptotically to the self-similar solution described in Ref. [30].

From Eq. (52) we see that when probability distribution function of initial potential decays very rapidly ($\alpha \to \infty$) the mean energy decays as t^{-1} just as it does for the stretched exponential type of initial potential. The special case where $\alpha = 1$ is equivalent to the uniform behavior of the probability distribution function for the initial potential over $H < H_M$, that is to say $W(H) = -\partial f(H)/\partial H = H_*^{-1}$. In this case the energy $E(t) \sim t^{2(1+d)/(2+d)}$ which is equivalent to the asymptotic law for the uniform distribution of the initial potential [28].

The limit $\alpha \to 0$ when the energy decays as t^{-2} in all dimension d is also interesting. In this case the probability distribution function of the coordinate (and of the velocity) has the universal form $W_s(z) \sim \exp(-z^d)$ for all z. It easy to see that for $\alpha \to 0$ the integral scale of the turbulence does not increase with time. The case of $\alpha \to 0$ has a simple explanation if we rewrite the cumulative probability distribution function for the initial potential corresponding in the form

$$f(H) = PE(H_M - H),$$

$$W(H) = -\frac{\partial f(H)}{\partial H} = P\delta(H - H_M). \qquad (56)$$

Here $E(z)$ is the unit function, $\delta(z)$ is the delta function, and P is the probability that in some cells the potential amplitude is exactly equal to H_M. This probability may by rewritten as $P = (L_0/L_P)^d$ where L_P is the typical distance between cells when the initial potential is equal to H_* and L_0 is the size of the cells. It is easy to see that in this case the universal behavior of energy decay $E(t) \sim t^{-2}$ is due to the fact that after some intermediate stage we have "frozen" turbulence. This means that the spatial structure of the velocity will not change in time and only the amplitude of the velocity will decay like $\Delta L/t$, where ΔL is the random distance between two cells with equal initial potential, i.e., $(H_i = H_j = H_M)$. In the case $d = 1$ the velocity will be a sequence of the lines $v = (x - x_i)/t$ with the immobile shocks at $(x_{i+1} - x_i)/2$, where x_i is the position of the point where $\psi_i = H_M$. This case was considered in Refs. [26,27] but with some other assumptions and with other tools. In arbitrary dimension d the energy decays as L_P^2/t^2, and the probability distribution function of the Lagrangian coordinate and velocity will have the stationary form $W(z) = Ae^{-z^d}$ with the scale $L \sim L_P \sim L_0 P^{-1/d}$. For $d = 2$ this distribution has a Gaussian form. In conclusion we note that this asymptotic "frozen" behavior for bounded initial perturbation of potential takes place for arbitrary α when $d \to \infty$.

C. Stretched exponential tail of the initial potential: Gumbel class

Let us now assume that the cumulative probability distribution function of the initial potential (25) has a stretched exponential tail

$$f(H) = (H/H_p)^{\alpha} \exp[-(H/H_*)^{\beta}]. \qquad (57)$$

For initial conditions of this type of the distribution potential is localized in a narrow region ΔH near the mean value $H_0 \gg H_*$. In the integral (30) over $\mathbf{y}$ we can take into account only the quadratic term in the exponent and obtain the following expression:

$$N(H,t) = \left(\frac{H}{H_p}\right)^{\alpha} \left[\frac{2\pi t H_*^{\beta}}{\beta H^{\beta-1} L_0^2}\right]^{d/2} e^{-(H/H_*)^{\beta}}. \qquad (58)$$

Let us introduce the dimensionless potential z so that

$$H = H_* h_0 \left(1 + \frac{z}{h_0^{\beta}\beta}\right), \qquad (59)$$

where h_0 is a solution of the transcendental equation $N(H_* h_0) = 1$:

$$h_0^{(2\alpha+d(1-\beta))/2} \left(\frac{t}{t_{nl}}\right)^{d/2} e^{-h_0^{\beta}} = 1,$$

$$h_0 \simeq \left(\frac{d}{2} \ln\frac{t}{t_{nl}}\right)^{1/\beta}. \qquad (60)$$

Here t_{nl} is the nonlinear time

$$t_{nl} = \frac{L_0^2}{2\pi H_*} \left(\frac{H_p}{H_*}\right)^{2\alpha/d}. \qquad (61)$$

The variable z in Eq. (59) has a universal double-exponential distribution at $t \gg t_{nl}$

$$F(z) = e^{-e^{-z}}. \qquad (62)$$

When $t \gg t_{nl}$ the distribution of the potential ψ is concentrated in the narrow region $\Delta H/H_0 \sim h_0^{-\beta} \sim [d \ln t/t_{nl}]^{-1}$ near the mean value $\langle H \rangle = H_* h_0 \sim (\ln t)^{1/\beta}$. From Eq. (21) we have the following expression for the energy of turbulence:

$$E(t) \simeq 2H_* \frac{\partial h_0}{\partial t} = \frac{2H_*(d/2)^{1/\beta}}{t\beta}(\ln t/t_{nl})^{(1-\beta)/\beta}. \qquad (63)$$

From Eq. (63) we see that the energy decays according to the universal law $E(t) \sim t^{-1}$ with some logarithmic correction: the decay is faster if $\beta > 1$ [tail of probability distribution function $f(H)$ decays faster then the exponential law], and slower if $\beta < 1$. It must be stressed also that the law of decay does not depend on the dimension of the space and so is the same as in one dimension [21].

The probability distribution function of absolute maximum coordinate **y** and the velocity **v** (7) are Gaussian and the variance of each component is given by

$$\langle y_i^2 \rangle = L^2(t) = \frac{tH_*}{\beta h_0^{\beta-1}} \sim \frac{t}{(\ln t/t_{nl})^{(\beta-1)/\beta}},$$

$$\langle v_i^2 \rangle = L^2/t^2. \qquad (64)$$

The two-point probability distributions of the velocity and correlation functions were found in Ref. [5,8] for the special case $\beta = 2$. The shape of the two-point probability distribution function of the longitudinal component is the same in the space of different dimensions, and coincides with the analogous PDF of the one-dimensional Burgers turbulence. The transverse velocity components, unlike the longitudinal ones, are statistically independent in different cells and have a Gaussian probability distribution inside them. These results may be extrapolated to the arbitrary stretched exponential tail of the initial potential.

V. CONCLUDING REMARKS

The present work has considered the evolution of random solutions of the unforced Burgers and equations in d dimensions in the limit of vanishing viscosity. The main statistical assumption is the independence of the initial velocity potential ψ_0 in different cells.

We show that the asymptotic behavior of the turbulence at large times is determined by the tail of cumulative initial potential probability distribution function $F(H) = 1 - f(H)$. We show that three classes of initial distribution lead to the self-similar evolution of the turbulence at large times. In the theory of extremes these limiting distributions are known as (a) Frechet class when $f(H) \propto H^{-\gamma}$; (b) Weibull class when $f(H) \propto (H_{\max} - H)^{-\gamma}, \gamma \leq 0$; and (c) Gumbel class when $f(H) \propto \exp(-H^\beta)$. One can find in Ref. [32] more general conditions which are necessary and sufficient for the probability distribution function $f(H)$ to belong to each of three types.

We show that the mean potential (mean height of the surface) increases with time in cases (a),(b) according to the power law $\langle \psi \rangle \propto At^r, r = d/(2\gamma - d)$ ($A < 0$ for $\gamma \leq 0$) while in case (c) it increases logarithmically according to $\langle \psi \rangle \propto (\ln t)^{1/\beta}$. For the Gumbel class the distribution of potential is localized in a narrow region near the mean value. For the Weibull class a PDF of potential exists for $\gamma > d/2$ and the mean value of ψ is finite for $\gamma > d/2 + 1$. The mean square gradient of the surface (turbulence energy) $E(t) = \langle [\nabla \psi(\mathbf{x},t)]^2 \rangle = \langle \mathbf{v}^2(\mathbf{x},t) \rangle$ has the power-law dependence $E(t) \propto t^{-p}, p = 2(\gamma - d)/(2\gamma - d)$ in cases (a) and (b) and decays according to $t^{-1}(\ln t)^{(1-\beta)/\beta}$ in case (c). For relatively slow decay of the initial probability distribution of potential $1 + d/2 < \gamma < d$ and dimension of the space $d > 2$ the energy $E(t)$ increases with time.

We show that the PDF of the velocity is self-similar with the scale $L(t)/t$, where $L(t)$ is the external scale of the turbulence. In cases (a) and (b) the external scale of turbulence increases according to the pure power law $L(t) \propto t^m, m = \gamma/(2\gamma - d)$ while in case (c) it increases like $L(t) \propto t^{1/2}(\ln t)^{(\beta-1)/\beta}$. One can see that for fast decaying initial distributions of potential ($|\gamma| \to \infty$) in classes (a) and (b) the law of external scale increase and the law of energy decay tends to the corresponding laws of the Gumbel class. In the special case $\gamma \to -0$ [class (b)] we have the "frozen" turbulence which means that the structures of the potential and velocity fields conserve $L(t) = \text{const}$ and the amplitude of the velocity increases as t^{-1}. We note that for the compactly supported potential this "frozen" behavior takes place for arbitrary γ when $d \to \infty$.

We now discuss what influence finite viscosity has on the asymptotic behavior of the Burgers turbulence at large times. For large initial Reynolds number ($\mathrm{Re}_0 \gg 1$) we still have the cellular structure of the turbulence at relatively large times. For finite Renolds number it will be characterized by two scales, namely the external scale $L(t)$ and inner scale δ the latter being a typical width of the shock surface [5]. Owing to viscosity the inner scale increases as $\delta \sim \nu t/L$. However due to the increase of $L(t)$ the relative width of the shock $\delta(t)/L(t) \sim \nu t/L^2(t) \sim \mathrm{Re}^{-1}(t)$ may either decrease or increase with time. Here we write the Reynolds number as $\mathrm{Re}(t) = V(t)L(t)/\nu = L^2(t)/\nu t$, since the local slope of the velocity is $1/t$ and the maximum velocity is of order $L(t)/t$.

It is easy to see that for classes (a) and (b) we have a power law for the Reynolds number, namely, $\mathrm{Re}(t) \propto t^r, r = d/(2\gamma - d)$. Thus for the Frechet class ($\gamma > d/2$) the Reynolds number increases with time and hence even when the viscosity is finite we have the strong nonlinear stage of evolution at large times. For the Weibull class ($\gamma \leq 0$) the Reynolds number decreases with time and at large times the evolution of the turbulence will be determined only by the linear diffusion. This is shown in Ref. [28] for the case of a uniform distribution of the initial potential. In Ref. [28] it was stressed that although one finds a universal power law growth for the energy decay (for a system with a bounded and flat initial distribution), the velocity-velocity correlation function has a more complicated form, and there is *no simple dynamical scaling in the system* in the "nonlinear regime." For the Gumbel class [$f(H) \propto \exp(-H^\beta)$] the Reynolds number $\mathrm{Re}(t) \propto (\ln t)^{(1-\beta)/\beta}$. Thus we may expect that for $\beta < 1$ we have conservation of the cellular structure at large times. For $\beta > 1$ the nonlinear evolution is only an intermediate asymptotic which changes at large times to linear decay. This effect is considered in Ref. [5,25] in the case of a Gaussian initial perturbation.

Let us now move to the case when the initial potential is

long-range correlated or nonhomogeneous. We consider first one-dimensional turbulence, assuming that the initial velocity is homogeneous with a spectrum $E_v(k) \sim \alpha^2 |k|^n$ at small wave numbers k and falling off quickly at large wave numbers. For the spectrum of potential we have $E_\psi(k) \sim \alpha^2 |k|^{n-2}$. For a Gaussian velocity it is shown in Ref. [35] that there are three regions of n with different behavior of turbulence. When $-1 < n < 1$ the long-time evolution of the spectrum is self-similar and the external scale of turbulence increases as $L(t) \propto (\alpha t)^m, m = 2/(3+n)$ and is determined by the ''amplitude'' of the large scale component α. When $1 < n < 2$ the spectrum has three scaling regions: first, a $|k|^n$ region at very small k with a time-independent constant, associated with long-correlated regions in physical space, second, a k^2 region at intermediate wave numbers which is related to the self-similarly evolving ''inner region'' in physical space and, finally, the usual k^{-2} region, associated to the shocks. The growth of the external scale is now determined by two integrals of the initial spectrum and $L(t) \propto t^{1/2} (\ln t)^{-1/4}$. Switching wave number from the $|k|^n$ to the k^2 region tends to zero faster than the energy wave number $\sim 1/L(t)$ and asymptotically we have the self-similar evolution of the spectrum. For $n > 2$, long-time evolution is also self-similar and $L(t) \propto t^{1/2} (\ln t)^{-1/4}$. Thus for a Gaussian perturbation we have one critical index n for the behavior of external scale and energy. When $n = 1$ the index of the power law dependence of $L(t)$ continuously transforms from $m = 2/(3+n), (n < 1)$ to the $m = 1/2$ index of the leading term in the region $n > 1$.

Let us assume that for the multidimensional Burgers turbulence the initial potential is isotropic and has a power-law dependence at small wave numbers, i.e., $E_\psi(k) \sim \alpha^2 |k|^{n-2}$. The variance of the potential is determined by $\langle \psi^2 \rangle \sim \int_0^\infty E_\psi(k) k^{d-1} dk$ and is finite when $n > 2 - d$. For $2 - d < n < -d$ its structure function is $d_\psi(\mathbf{x}) = \langle [\psi_0(\mathbf{x}) - \psi_0(0)]^2 \rangle \sim |\mathbf{x}|^{2-n-d}$. Using the rescaling of the structure function in Eq. (6) we see that the external scale increases as $L(t) \sim (\alpha t)^m, m = 2/(2+n+d)$ [5,11]. This law does not depend on the PDF of the initial potential. Introducing the Reynolds number directly through the Hopf-Cole solution [5] $\text{Re}(t) = L^2(t)/\nu = V(t)L(t)/\nu t$ we see that in this region of n the Reynolds number increases with time according to $\propto t^{(2-n-d)/(2+n+d)}$. It means that at late stage the turbulence has a strong nonlinear cellular structure.

Considering the case of independent amplitudes in different cells, we see that the correlation function of the initial potentials equal to zero if the distance between the points is greater then the size of the cell. Thus, the discrete model of the initial conditions considered in the present paper is equal to the uncorrelated potential with $n \geq 2$. Here we show that the laws of evolution of the external scale $L(t)$ (48),(53),(64) and the energy $E(t)$ (43),(52),(63) are very sensitive to the tail of the PDF of the initial potential. For the stretched exponential tail (i.e., the Gumbel class) the leading term for the evolution of the external scale is $t^{1/2}$. Thus we can expect that in multidimensional turbulence the long-range correlation of the potential for $2 - d < n < 2$ does not play an important role for the evolution of the external scale and energy and we have only one critical index n. This is shown in Refs. [24,35] for the one-dimensional case with a Gaussian potential. When $n = 2 - d$ the universal index of the power law dependence of $L(t)$ continuously transforms from $m = 2/(2+n+d), (n < 2 - d)$ to the index $m = 1/2$ for the leading term in the region where $n > 2 - d$. In this case we have also the same critical index $n = 2 - d$ for the energy. In the interval $2 - d < n < 2$ we have conservation of the velocity spectrum $E_v(k) \propto |k|^n$ at very small wave numbers. But this small region is not significant for the energy of the turbulence and asymptotically the spectrum of the velocity tends to the self-similar evolution.

We have a much more nontrivial situation when the potential distribution has a with power-law tail (Frechet class) or when it is a compactly supported potential (Weibull class). For $n < 2 - d$ we find, using the scaling properties of solution (6), that the external scale increases as $L(t) \sim (\alpha t)^m, m = 2/(2+n+d)$. Thus the evolution of $L(t)$ and the energy $E(t) \sim L^2(t)/t^2$ do not depend on the PDF of the initial potential. For the uncorrelated potential $n \geq 2$ the power indexes laws of the scale evolution $m = \gamma/(2\gamma - d)$ and energy evolution $p = 2(\gamma - d)/(2\gamma - d)$ depend on γ. The case of a homogeneous continuous potential with a power index $2 - d < n < 2$ is equal to the existence of long-range correlation of the the potential amplitudes in cells. We can expect that in the region $2 - d < n < 2$ the long-range correlation of potential influences the energy decay.

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